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Differential Sandwich Theorem for Multivalent Meromorphic Functions associated with the Liu-Srivastava Operator

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Abstract. Differential subordination and superordination results are obtained for multivalent meromorphic functions associated with the Liu-Srivastava linear operator in the punctured unit disk. These results are derived by investigating appropriate classes of admissible functions. Sandwich-type results are also obtained.

Dedicated to Professor H. M. Srivastava on the occasion of his 70th birthday.

1. Introduction

Let $\mathcal{H}(U)$ be the class of functions analytic in $U := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$, with $\mathcal{H} \equiv \mathcal{H}[1, 1]$. Let Σ_p denote the class of all meromorphic p-valent functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \geq 1) \tag{1.1}$$

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that are analytic in the punctured open unit disk $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$, and let $\Sigma_1 := \Sigma$. Let f and F be members of $\mathcal{H}(U)$. A function f is said to be *subordinate* to F, or F is said to be *superordinate* to f, written $f(z) \prec F(z)$, if there exists a function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ satisfying $f(z) = F(w(z))$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$. For two functions $f, g \in \Sigma_p$, where f is given by (1.1) and $g(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by the series

$$(f * g)(z) := \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k b_k z^k =: (g * f)(z).$$

For $\alpha_j \in \mathbb{C}$ $\quad (j = 1, 2, \ldots, l)$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ $(j = 1, 2, \ldots, m)$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_l)_k}{(\beta_1)_k \ldots (\beta_m)_k} \frac{z^k}{k!}$$

$$(l \leq m + 1; l, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}),$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a+1)(a+2)\ldots(a+n-1), & (n \in \mathbb{N} := \{1, 2, 3 \ldots\}). \end{cases}$$

Corresponding to the function

$$h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z^{-p} \, {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

the Liu-Srivastava operator [17, 18] $\widetilde{H}_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) : \Sigma_p \to \Sigma_p$ is defined by the Hadamard product

$$\begin{aligned} \widetilde{H}_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) &:= h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \\ &= \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \frac{(\alpha_1)_{k+p} \ldots (\alpha_l)_{k+p}}{(\beta_1)_{k+p} \ldots (\beta_m)_{k+p}} \frac{a_k z^k}{(k+p)!}. \end{aligned} \tag{1.2}$$

For convenience, (1.2) is written as

$$\widetilde{H}_p^{l,m}[\beta_1]f(z) := \widetilde{H}_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z).$$

Various authors have investigated the Liu-Srivastava operator where (1.2) is framed using the notation $H_p^{l,m}[\alpha_1]f(z)$. Their works exploited the recurrence relation involving the parameter α_1 in the numerator satisfying

$$\alpha_1 H_p^{l,m}[\alpha_1 + 1]f(z) = z[H_p^{l,m}[\alpha_1]f(z)]' + (\alpha_1 + p)H_p^{l,m}[\alpha_1]f(z).$$

The parameter β_1 in the denominator also satisfies a recurrence relation. Denoting (1.2) by $\widetilde{H}_p^{l,m}[\beta_1]f(z)$, it can be shown that

$$\beta_1 \widetilde{H}_p^{l,m}[\beta_1]f(z) = z[\widetilde{H}_p^{l,m}[\beta_1+1]f(z)]' + (\beta_1+p)\widetilde{H}_p^{l,m}[\beta_1+1]f(z). \tag{1.3}$$

The analytic analogue of the Liu-Srivastava operator known as the Dziok-Srivastava operator with respect to the parameter β_1, was first investigated by Srivastava *et al.* [24], and more recently by Ali *et al.* [3]. However, the Liu-Srivastava operator for the parameter β_1 given by (1.3) for functions f satisfying (1.1) seems yet to be investigated. Special cases of the Liu-Srivastava linear operator include the meromorphic analogue of the Carlson-Shaffer linear operator $\mathcal{L}_p(a,c) := \widetilde{H}_p^{(2,1)}(1,a;c)$[14, 16, 27], the operator $D^{n+1} := \mathcal{L}_p(n+p,1)$, which is analogous to the Ruscheweyh derivative operator [26], and the operator

$$J_{c,p} := \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t)dt = \mathcal{L}_p(c, c+1) \quad (c>0)$$

studied by Uralegaddi and Somanatha [25]. It is clear that the Liu-Srivastava operator investigated in [11, 22, 23] is the meromorphic analogue of the Dziok-Srivastava [12] linear operator.

To state our main results, the following definitions and theorems will be required.

Denote by $\mathcal{Q}$ the set of all functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where

$$E(q) = \{\zeta \in \partial U : \lim_{z\to\zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

Definition 1.1([19, Definition 2.3a, p. 27]**).** Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfying the admissibility condition $\psi(r,s,t;z) \notin \Omega$ whenever $r = q(\zeta)$, $s = k\zeta q'(\zeta)$, and

$$\operatorname{Re}\left(\frac{t}{s}+1\right) \geq k \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right),$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

In particular when $q(z) = M\frac{Mz+a}{M+\bar{a}z}$, with $M > 0$ and $|a| < M$, then $q(U) = U_M := \{w : |w| < M\}$, $q(0) = a$, $E(q) = \emptyset$ and $q \in \mathcal{Q}(a)$. In this case, we set $\Psi_n[\Omega, q] := \Psi_n[\Omega, M, a]$, and in the special case when the set $\Omega = U_M$, the class is simply denoted by $\Psi_n[M, a]$.

Definition 1.2.([20, Definition 3, p. 817]**).** Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}[a,n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions

$\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ satisfying the admissibility condition $\psi(r, s, t; \zeta) \in \Omega$ whenever $r = q(z)$, $s = \frac{zq'(z)}{m}$, and

$$\operatorname{Re}\left(\frac{t}{s} + 1\right) \leq \frac{1}{m} \operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

$z \in U$, $\zeta \in \partial U$ and $m \geq n \geq 1$. In particular, we write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

For the above two classes of admissible functions, Miller and Mocanu [19, 20] proved the following theorems.

Theorem 1.1([19, Theorem 2.3b, p. 28])**.** *Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If the analytic function $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$ satisfies*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega,$$

then $p(z) \prec q(z)$.

Theorem 1.2([20, Theorem 1, p. 818])**.** *Let $\psi \in \Psi'_n[\Omega, q]$ with $q(0) = a$. If $p \in \mathcal{Q}(a)$ and $\psi(p(z), zp'(z), z^2p''(z); z)$ is univalent in U, then*

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z) : z \in U\}$$

implies $q(z) \prec p(z)$.

In the present investigation, the differential subordination results of Miller and Mocanu [19, Theorem 2.3b, p. 28] are extended for functions associated with the Liu-Srivastava linear operator $\widetilde{H}_p^{l,m}[\beta_1]$. A similar problem was first studied by Aghalary *et al.* [1, 2], and related results may be found in the works of [4, 5, 6, 7, 8, 9, 10, 13, 15]. Additionally, the corresponding differential superordination problem is also investigated, and several sandwich-type results are obtained. Analogous results for analytic functions in the class associated with the Dziok-Srivastava operator can be found in [3].

2. Subordination results involving the Liu-Srivastava linear operator

The following class of admissible functions will be required in the first result.

Definition 2.1. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_H[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfying the admissibility condition

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) + (\beta_1 + 1)q(\zeta)}{\beta_1 + 1} \qquad (\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}),$$

$$\operatorname{Re}\left(\frac{\beta_1(w - u)}{(v - u)} - (2\beta_1 + 1)\right) \geq k \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq 1$.

Choosing $q(z) = 1 + Mz,\ M > 0$, Definition 2.1 easily gives the following definition.

Definition 2.2. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_H[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ such that

$$\phi\left(1 + Me^{i\theta}, 1 + \frac{k+\beta_1+1}{\beta_1+1}Me^{i\theta}, 1 + \frac{L + (\beta_1+1)(2k+\beta_1)Me^{i\theta}}{\beta_1(\beta_1+1)}; z\right) \notin \Omega \tag{2.1}$$

whenever $z \in U$, $\theta \in \mathbb{R}$, $\mathrm{Re}(Le^{-i\theta}) \geq (k-1)kM$ for all real θ, $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ and $k \geq 1$.

In the special case $\Omega = q(U) = \{\omega : |\omega - 1| < M\}$, the class $\Phi_H[\Omega, M]$ is simply denoted by $\Phi_H[M]$.

Theorem 2.1. *Let $\phi \in \Phi_H[\Omega, q]$. If $f \in \Sigma_p$ satisfies*

$$\left\{\phi\left(z^p \widetilde{H}_p^{l,m}[\beta_1+2]f(z), z^p \widetilde{H}_p^{l,m}[\beta_1+1]f(z), z^p \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right) : z \in U\right\} \subset \Omega, \tag{2.2}$$

then

$$z^p \widetilde{H}_p^{l,m}[\beta_1+2]f(z) \prec q(z), \qquad (z \in U).$$

Proof. Define the analytic function p in U by

$$p(z) := z^p \widetilde{H}_p^{l,m}[\beta_1+2]f(z). \tag{2.3}$$

In view of the relation (1.3), it follows from (2.3) that

$$z^p \widetilde{H}_p^{l,m}[\beta_1+1]f(z) = \frac{1}{\beta_1+1}[(\beta_1+1)p(z) + zp'(z)]. \tag{2.4}$$

Further computations show that

$$z^p \widetilde{H}_p^{l,m}[\beta_1]f(z) = \frac{1}{\beta_1(\beta_1+1)}[z^2p''(z) + 2(\beta_1+1)zp'(z)] + p(z). \tag{2.5}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r,\ v = \frac{s + (\beta_1+1)r}{\beta_1+1},\ w = \frac{t + 2(\beta_1+1)s + (\beta_1)(\beta_1+1)r}{\beta_1(\beta_1+1)}. \tag{2.6}$$

Let

$$\begin{aligned} \psi(r,s,t;z) &= \phi(u,v,w;z) \\ &= \phi\left(r, \frac{s+(\beta_1+1)r}{\beta_1+1}, \frac{t + 2(\beta_1+1)s + (\beta_1)(\beta_1+1)r}{\beta_1(\beta_1+1)}; z\right). \end{aligned} \tag{2.7}$$

From (2.3), (2.4) and (2.5), Equation (2.7) yields

$$\begin{aligned}(2.8)\qquad &\psi(p(z), zp'(z), z^2p''(z); z)\\ &= \phi\left(z^p\widetilde{H}_p^{l,m}[\beta_1+2]f(z), z^p\widetilde{H}_p^{l,m}[\beta_1+1]f(z), z^p\widetilde{H}_p^{l,m}[\beta_1]f(z); z\right).\end{aligned}$$

Hence (2.2) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

To complete the proof , it is left to show that the admissibility condition for $\phi \in \Phi_H[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. Note that

$$\frac{t}{s} + 1 = \frac{\beta_1(w-u)}{(v-u)} - (2\beta_1 + 1),$$

and hence $\psi \in \Psi_n[\Omega, q]$.

By Theorem 1.1, $p(z) \prec q(z)$ or $z^p\widetilde{H}_p^{l,m}[\beta_1+2]f(z) \prec q(z)$. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping h of U onto Ω. In this case the class $\Phi_H[h(U), q]$ is written as $\Phi_H[h, q]$. The following result is an immediate consequence of Theorem 2.1.

Theorem 2.2. *Let $\phi \in \Phi_H[h, q]$ with $q(0) = 1$. If $f \in \Sigma_p$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ satisfies*

$$(2.9)\qquad \phi\left(z^p\widetilde{H}_p^{l,m}[\beta_1+2]f(z), z^p\widetilde{H}_p^{l,m}[\beta_1+1]f(z), z^p\widetilde{H}_p^{l,m}[\beta_1]f(z); z\right) \prec h(z),$$

then

$$z^p\widetilde{H}_p^{l,m}[\beta_1+2]f(z) \prec q(z).$$

The next result is an extension of Theorem 2.1 to the case where the behavior of q on ∂U is not known.

Corollary 2.1. *Let $\Omega \subset \mathbb{C}$, q be univalent in U and $q(0) = 1$. Let $\phi \in \Phi_H[\Omega, q_\rho]$ for some $\rho \in (0,1)$ where $q_\rho(z) = q(\rho z)$. If $f \in \Sigma_p$ and*

$$\phi\left(z^p\widetilde{H}_p^{l,m}[\beta_1+2]f(z), z^p\widetilde{H}_p^{l,m}[\beta_1+1]f(z), z^p\widetilde{H}_p^{l,m}[\beta_1]f(z); z\right) \in \Omega,$$

then

$$z^p\widetilde{H}_p^{l,m}[\beta_1+2]f(z) \prec q(z).$$

Proof. Theorem 2.1 yields $z^p\widetilde{H}_p^{l,m}[\beta_1+2]f(z) \prec q_\rho(z)$. The result now follows from the fact that $q_\rho(z) \prec q(z)$. □

Theorem 2.3. *Let h and q be univalent in U, with $q(0) = 1$, and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy one of the following conditions:*

1. $\phi \in \Phi_H[h, q_\rho]$ *for some* $\rho \in (0,1)$, *or*
2. *there exists* $\rho_0 \in (0,1)$ *such that* $\phi \in \Phi_H[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_0, 1)$.

If $f \in \Sigma_p$ *satisfies* (2.9), then $z^p \widetilde{H}_p^{l,m}[\beta_1 + 2]f(z) \prec q(z)$.

Proof. The result is similar to the proof of Theorem 2.3d [19, p. 30] and is omitted. □

The next theorem yields the best dominant of the differential subordination (2.9).

Theorem 2.4. *Let* h *be univalent in* U, *and* $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$. *Suppose that the differential equation*

$$\phi\left(q(z), zq'(z), z^2 q''(z); z\right) = h(z) \tag{2.10}$$

has a solution q *with* $q(0) = 1$ *and satisfy one of the following conditions:*

1. $q \in \mathcal{Q}_1$ *and* $\phi \in \Phi_H[h, q]$,
2. q *is univalent in* U *and* $\phi \in \Phi_H[h, q_\rho]$ *for some* $\rho \in (0,1)$, *or*
3. q *is univalent in* U *and there exists* $\rho_0 \in (0,1)$ *such that* $\phi \in \Phi_H[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_0, 1)$.

If $f \in \Sigma_p$ *satisfies* (2.9), then

$$z^p \widetilde{H}_p^{l,m}[\beta_1 + 2]f(z) \prec q(z),$$

and q is the best dominant.

Proof. Following the same arguments in [19, Theorem 2.3e, p. 31], the function q is a dominant from Theorems 2.2 and 2.3. Since q satisfies (2.10), it is also a solution of (2.9) and therefore q will be dominated by all dominants. Hence q is the best dominant. □

Corollary 2.2. *Let* $\phi \in \Phi_H[\Omega, M]$. *If* $f \in \Sigma_p$ *satisfies*

$$\phi\left(z^p \widetilde{H}_p^{l,m}[\beta_1 + 2]f(z), z^p \widetilde{H}_p^{l,m}[\beta_1 + 1]f(z), z^p \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right) \in \Omega,$$

then

$$\left|z^p \widetilde{H}_p^{l,m}[\beta_1 + 2]f(z) - 1\right| < M.$$

Corollary 2.3. *Let* $\phi \in \Phi_H[M]$. *If* $f \in \Sigma_p$ *satisfies*

$$\left|\phi\left(z^p \widetilde{H}_p^{l,m}[\beta_1 + 2]f(z), z^p \widetilde{H}_p^{l,m}[\beta_1 + 1]f(z), z^p \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right) - 1\right| < M,$$

then

$$\left|z^p \widetilde{H}_p^{l,m}[\beta_1 + 2]f(z) - 1\right| < M.$$

The following example is easily obtained by taking $\phi(u,v,w;z) = v$ in Corollary 2.3.

Example 2.1. If $\operatorname{Re}\beta_1 \geq -\left(\frac{k}{2}+1\right)$ and $f \in \Sigma_p$ satisfies

$$\left|z^p \widetilde{H}_p^{l,m}[\beta_1+1]f(z) - 1\right| < M,$$

then

$$\left|z^p \widetilde{H}_p^{l,m}[\beta_1+l]f(z) - 1\right| < M$$

for $l = 2, 3, \ldots$.

Corollary 2.4. *Let* $M > 0$. *If* $f \in \Sigma_p$ *and*

$$\left|z^p \widetilde{H}_p^{l,m}[\beta_1+1]f(z) - z^p \widetilde{H}_p^{l,m}[\beta_1+2]f(z)\right| < \frac{M}{|\beta_1+1|},$$

then

$$\left|z^p \widetilde{H}_p^{l,m}[\beta_1+2]f(z) - 1\right| < M.$$

Proof. Let $\phi(u,v,w;z) = v - u$ and $\Omega = h(U)$ where $h(z) = \frac{M}{|\beta_1+1|}z$, $M > 0$. It suffices to show that $\phi \in \Phi_H[\Omega, M]$, that is, the admissible condition (2.1) is satisfied. This follows since

$$\begin{aligned}&\left|\phi\left(1+Me^{i\theta}, 1+\frac{k+\beta_1+1}{\beta_1+1}Me^{i\theta}, 1+\frac{L+(\beta_1+1)(2k+\beta_1)Me^{i\theta}}{\beta_1(\beta_1+1)}; z\right)\right| \\ &= \frac{kM}{|\beta_1+1|} \geq \frac{M}{|\beta_1+1|},\end{aligned}$$

$z \in U$, $\theta \in \mathbb{R}$, $\beta_1 \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}$ and $k \geq 1$. From Corollary 2.2, the required result is obtained. □

Definition 2.3. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_{H,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfying the admissibility condition

$$\phi(u,v,w;z) \notin \Omega$$

whenever

$$u = q(\zeta),\ v = \frac{(\beta_1+1)q(\zeta)}{(\beta_1+2) - k\zeta q'(\zeta) - q(\zeta)}, \quad (\beta_1 \in \mathbb{C}\setminus\{0,-1,-2,\ldots\}, \quad q(\zeta) \neq 0),$$

$$\operatorname{Re}\left(\frac{(\beta+1)u}{v(\beta+2)-(\beta+1)u-vu}\left[\frac{\beta+1}{v} - \frac{\beta}{w} - 1\right] - \frac{\beta+1}{v} - 1\right) \geq k \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq 1$.

In the particular case $q(z) = 1 + Mz,\ M > 0$, Definition 2.3 yields the following definition.

Definition 2.4. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{H,1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfying

$$\phi\Bigg(1 + Me^{i\theta}, \frac{(\beta_1+1)(1+Me^{i\theta})}{(\beta_1+1) - Me^{i\theta}(k+1)}\ , \\ \frac{\beta_1(1+Me^{i\theta})[\beta_1+1-Me^{i\theta}(k+1)]}{[\beta_1+1-Me^{i\theta}(k+1)][\beta_1-2Me^{i\theta}(k+1)] - (1+Me^{i\theta})(L+2kMe^{i\theta})}; z\Bigg) \notin \Omega$$

whenever $z \in U,\ \theta \in \mathbb{R},\ \mathrm{Re}(Le^{-i\theta}) \geq (k-1)kM$ for all real θ, $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ and $k \geq 1$.

In the special case $\Omega = q(U) = \{\omega : |\omega - 1| < M\}$, the class $\Phi_{H,1}[\Omega, M]$ is simply denoted by $\Phi_{H,1}[M]$.

Theorem 2.5. *Let $\phi \in \Phi_{H,1}[\Omega, q]$. If $f \in \Sigma_p$ satisfies*

$$\left\{\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)}; z\right) : z \in U\right\} \subset \Omega, \tag{2.11}$$

then

$$\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} \prec q(z).$$

Proof. Define the analytic function p in U by

$$p(z) := \frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}. \tag{2.12}$$

This implies

$$\frac{zp'(z)}{p(z)} := \frac{z[\widetilde{H}_p^{l,m}[\beta_1+3]f(z)]'}{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)} - \frac{z[\widetilde{H}_p^{l,m}[\beta_1+2]f(z)]'}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)},$$

which from (1.3) yields

$$\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)} = \frac{(\beta_1+1)p(z)}{\beta_1+2-zp'(z)-p(z)}. \tag{2.13}$$

Further computations show that

$$\frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)} = \frac{\beta_1}{\frac{\beta_1-2-zp'(z)-p(z)}{p(z)} - \frac{zp'(z)}{p(z)} - \frac{[z^2p''(z)+2zp'(z)]}{\beta_1-2-zp'(z)-p(z)} - 1}. \tag{2.14}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \; v = \frac{(\beta_1+1)r}{\beta_1+2-s-r}, \; w = \frac{\beta_1}{\frac{\beta_1+2-s-r}{r} - \frac{s}{r} - \frac{(t+2s)}{\beta_1+2-s-r} - 1}. \tag{2.15}$$

Let

$$\begin{aligned} \psi(r,s,t;z) &:= \phi(u,v,w;z) \\ &= \phi\left(r, \frac{(\beta_1+1)r}{\beta_1+2-s-r}, \frac{\beta_1}{\frac{\beta_1+2-s-r}{r} - \frac{s}{r} - \frac{(t+2s)}{\beta_1+2-s-r} - 1}; z\right). \end{aligned} \tag{2.16}$$

The proof shall make use of Theorem 1.1. Using equations (2.12), (2.13) and (2.14) in (2.16) yield

$$\begin{aligned} &\psi(p(z), zp'(z), z^2p''(z); z) \\ &= \phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)}; z\right). \end{aligned} \tag{2.17}$$

Hence (2.11) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

To complete the proof, the admissibility condition for $\phi \in \Phi_{H,2}[\Omega, q]$ is shown to be equivalent to the admissibility condition for ψ as given in Definition 1.1. Note that

$$\frac{t}{s} + 1 = \operatorname{Re}\left(\frac{(\beta+1)u}{v(\beta+2) - (\beta+1)u - vu}\left[\frac{\beta+1}{v} - \frac{\beta}{w} - 1\right] - \frac{\beta+1}{v} - 1\right),$$

and hence $\psi \in \Psi[\Omega, q]$. By Theorem 1.1, $p(z) \prec q(z)$ or

$$\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} \prec q(z). \qquad \square$$

As in the previous cases, if $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping h of U onto Ω. In this case the class $\Phi_{H,1}[h(U), q]$ is written as $\Phi_{H,1}[h, q]$. The following result is an immediate consequence of Theorem 2.5.

Theorem 2.6. *Let $\phi \in \Phi_{H,1}[h, q]$. If $f \in \Sigma_p$ satisfies*

$$\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)}; z\right) \prec h(z),$$

then

$$\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} \prec q(z).$$

Corollary 2.5. *Let* $\phi \in \Phi_{H,1}[\Omega, M]$. *If* $f \in \Sigma_p$ *satisfies*

$$\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)}; z\right) \in \Omega,$$

then

$$\left|\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} - 1\right| < M.$$

Corollary 2.6. *Let* $\phi \in \Phi_{H,1}[M]$. *If* $f \in \Sigma_p$ *satisfies*

$$\left|\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)}; z\right) - 1\right| < M,$$

then

$$\left|\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} - 1\right| < M.$$

3. Superordination of the Liu-Srivastava linear operator

The dual problem of differential subordination, that is, differential superordination of the Liu-Srivastava linear operator is investigated in this section. For this purpose, the following class of admissible functions will be required.

Definition 3.1. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_H[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ satisfying the admissibility condition

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z), \quad v = \frac{zq'(z) + m(\beta_1+1)q(z)}{m(\beta_1+1)} \quad (\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}),$$

$$\operatorname{Re}\left(\frac{\beta_1(w-u)}{(v-u)} - (2\beta_1+1)\right) \leq \frac{1}{m}\operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

$z \in U$, $\zeta \in \partial U$ and $m \geq 1$.

Theorem 3.1. *Let* $\phi \in \Phi'_H[\Omega, q]$. *If* $f \in \Sigma_p$, $z^p \widetilde{H}_p^{l,m}[\beta_1+2]f(z) \in \mathcal{Q}_1$ *and*

$$\phi\left(z^p \widetilde{H}_p^{l,m}[\beta_1+2]f(z), z^p \widetilde{H}_p^{l,m}[\beta_1+1]f(z), z^p \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right)$$

is univalent in U, *then*

$$\Omega\subset\left\{\phi\left(z^p\widetilde{H}_p^{l,m}[\beta_1+2]f(z),z^p\widetilde{H}_p^{l,m}[\beta_1+1]f(z),z^p\widetilde{H}_p^{l,m}[\beta_1]f(z);z\right):z\in U\right\} \tag{3.1}$$

implies

$$q(z)\prec z^p\widetilde{H}_p^{l,m}[\beta_1+2]f(z).$$

Proof. From (2.8) and (3.1), it follows that

$$\Omega\subset\left\{\psi\left(p(z),zp'(z),z^2p''(z);z\right):z\in U\right\}.$$

From (2.6), it is clear that the admissibility condition for $\phi\in\Phi'_H[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence $\psi\in\Psi'_p[\Omega,q]$, and by Theorem 1.2, $q(z)\prec p(z)$ or

$$q(z)\prec z^p\widetilde{H}_p^{l,m}[\beta_1+2]f(z). \qquad \square$$

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(U)$ for some conformal mapping h of U onto Ω. In this case the class $\Phi'_H[h(U),q]$ is written as $\Phi'_H[h,q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.1.

Theorem 3.2. *Let* $q\in\mathcal{H}$, h *be analytic in* U *and* $\phi\in\Phi'_H[h,q]$. *If* $f\in\Sigma_p$, $z^p\widetilde{H}_p^{l,m}[\beta_1+2]f(z)\in\mathcal{Q}_1$ *and* $\phi\left(z^p\widetilde{H}_p^{l,m}[\beta_1+2]f(z),z^p\widetilde{H}_p^{l,m}[\beta_1+1]f(z),z^p\widetilde{H}_p^{l,m}[\beta_1]f(z);z\right)$ *is univalent in* U, *then*

$$h(z)\prec\phi\left(z^p\widetilde{H}_p^{l,m}[\beta_1+2]f(z),z^p\widetilde{H}_p^{l,m}[\beta_1+1]f(z),z^p\widetilde{H}_p^{l,m}[\beta_1]f(z);z\right) \tag{3.2}$$

implies

$$q(z)\prec z^p\widetilde{H}_p^{l,m}[\beta_1+2]f(z).$$

Theorem 3.1 and 3.2 can only be used to obtain subordinants of differential superordination of the form (3.1) or (3.2). The following theorem proves the existence of the best subordinant of (3.2) for certain ϕ.

Theorem 3.3. *Let* h *be analytic in* U *and* $\phi:\mathbb{C}^3\times\overline{U}\to\mathbb{C}$. *Suppose that the differential equation*

$$\phi(q(z),zq'(z),z^2q''(z);z)=h(z)$$

has a solution $q\in\mathcal{Q}_1$. *If* $\phi\in\Phi'_H[h,q]$, $f\in\Sigma_p$, $z^p\widetilde{H}_p^{l,m}[\beta_1+2]f(z)\in\mathcal{Q}_1$ *and*

$$\phi\left(z^p\widetilde{H}_p^{l,m}[\beta_1+2]f(z),z^p\widetilde{H}_p^{l,m}[\beta_1+1]f(z),z^p\widetilde{H}_p^{l,m}[\beta_1]f(z);z\right)$$

is univalent in U, *then*

$$h(z) \prec \phi\left(z^p \widetilde{H}_p^{l,m}[\beta_1+2]f(z), z^p \widetilde{H}_p^{l,m}[\beta_1+1]f(z), z^p \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right)$$

implies

$$q(z) \prec z^p \widetilde{H}_p^{l,m}[\beta_1+2]f(z),$$

and q *is the best subordinant.*

Proof. The proof is similar to the proof of Theorem 2.4 and is therefore omitted. □

Combining Theorems 2.2 and 3.2, we obtain the following sandwich-type theorem.

Corollary 3.1. *Let* h_1 *and* q_1 *be analytic functions in* U, h_2 *be univalent function in* U, $q_2 \in \mathcal{Q}_1$ *with* $q_1(0) = q_2(0) = 1$ *and* $\phi \in \Phi_H[h_2, q_2] \cap \Phi'_H[h_1, q_1]$. *If* $f \in \Sigma_p$, $z^p \widetilde{H}_p^{l,m}[\beta_1+2]f(z) \in \mathcal{H} \cap \mathcal{Q}_1$ *and*

$$\phi\left(z^p \widetilde{H}_p^{l,m}[\beta_1+2]f(z), z^p \widetilde{H}_p^{l,m}[\beta_1+1]f(z), z^p \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right)$$

is univalent in U, *then*

$$h_1(z) \prec \phi\left(z^p \widetilde{H}_p^{l,m}[\beta_1+2]f(z), z^p \widetilde{H}_p^{l,m}[\beta_1+1]f(z), z^p \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right) \prec h_2(z)$$

implies

$$q_1(z) \prec z^p \widetilde{H}_p^{l,m}[\beta_1+2]f(z) \prec q_2(z).$$

Definition 3.2. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{H,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ satisfying the admissibility condition

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z),\ v = \frac{m(\beta_1+1)q(z)}{m(\beta_1+2) - zq'(z) - mq(z)}, \quad (\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}),$$

$$\operatorname{Re}\left(\frac{(\beta+1)u}{v(\beta+2)-(\beta+1)u-vu}\left[\frac{\beta+1}{v} - \frac{\beta}{w} - 1\right] - \frac{\beta+1}{v} - 1\right) \leq \frac{1}{m} \operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

$z \in U$, $\zeta \in \partial U$ and $m \geq 1$.

Next the dual result of Theorem 2.5 for differential superordination is given.

Theorem 3.4. *Let* $\phi \in \Phi'_{H,1}[\Omega, q]$. *If* $f \in \Sigma_p$, $\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} \in \mathcal{Q}_1$ *and*

$$\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)}; z\right)$$

is univalent in U, *then*

$$(3.3)\quad \Omega\subset\left\{\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)},\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)},\frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)};z\right):z\in U\right\}$$

implies

$$q(z)\prec\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}.$$

Proof. From (2.17) and (3.3), it follows that

$$\Omega\subset\left\{\phi\left(p(z),zp'(z),z^2p''(z);z\right):z\in U\right\}.$$

From (2.15), the admissibility condition for $\phi\in\Phi'_{H,2}[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence $\psi\in\Psi'[\Omega,q]$, and by Theorem 1.2, $q(z)\prec p(z)$ or

$$q(z)\prec\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}.\qquad\square$$

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(U)$ for some conformal mapping h of U onto Ω. In this case the class $\Phi'_{H,1}[h(U),q]$ is written as $\Phi'_{H,1}[h,q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.4.

Theorem 3.5. *Let* $q\in\mathcal{H}$, h *be analytic in* U *and* $\phi\in\Phi'_{H,1}[h,q]$. *If* $f\in\Sigma_p$, $\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}\in\mathcal{Q}_1$ *and* $\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)},\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)},\frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)};z\right)$ *is univalent in* U, *then*

$$h(z)\prec\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)},\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)},\frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)};z\right)$$

implies

$$q(z)\prec\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}.$$

Combining Theorems 2.6 and 3.5, the following sandwich-type theorem is obtained.

Corollary 3.2. *Let* h_1 *and* q_1 *be analytic functions in* U, h_2 *be univalent function in* U, $q_2\in\mathcal{Q}_1$ *with* $q_1(0)=q_2(0)=1$ *and* $\phi\in\Phi_{H,1}[h_2,q_2]\cap\Phi'_{H,1}[h_1,q_1]$. *If* $f\in\Sigma_p$, $\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}\in\mathcal{H}\cap\mathcal{Q}_1$ *and*

$$\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)},\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)},\frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)};z\right)$$

is univalent in U, *then*

$$h_1(z) \prec \phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)}; z\right) \prec h_2(z)$$

implies

$$q_1(z) \prec \frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} \prec q_2(z).$$

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